Regular Polygons

ACTIVITY 31

Plenty of Polygons

Lesson 31-1 Sum of the Measures of the Interior Angles of a Polygon

My Notes

Learning Targets:

- Develop a formula for the sum of the measures of the interior angles of a polygon.
- Determine the sum of the measures of the interior angles of a polygon.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Create Representations, Look for a Pattern, Self Revision/Peer Revision

A ***polygon*** is a closed geometric figure with sides formed by three or more coplanar segments that intersect exactly two other segments, one at each endpoint. The angles formed inside the polygon are ***interior angles***. Although it is difficult to measure lengths and angles exactly, tools such as rulers and protractors allow you to measure with reasonable accuracy.

Work with your group and use the polygons on the next two pages.

1. **Use appropriate tools strategically.** Measure, as precisely as possible, each interior angle, and record your results below. Complete the table by calculating the indicated sums.

	1st Angle	2nd Angle	3rd Angle	4th Angle	5th Angle	6th Angle	Total
Triangle							
Quadrilateral							
Pentagon							

TECHNOLOGY TIP

You can draw the polygons using geometry software. Then use the software to determine the angle measures and complete the table.

2. Compare your results in the table with those of other groups in your class. What similarities do you notice?

3. Write a conjecture about the sum of the measures of the interior angles of each of the polygons.

Triangle:

The sum of the measures is 180°

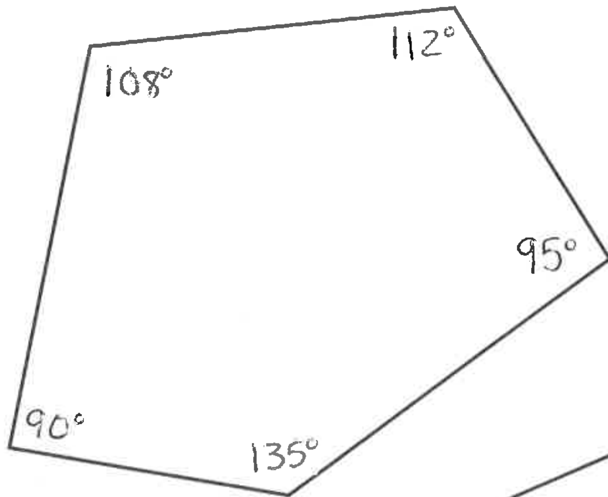
Quadrilateral:

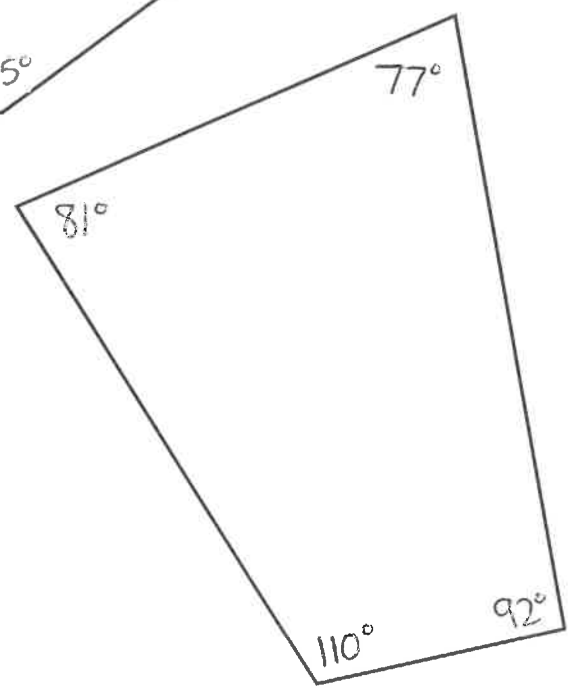
The sum of the measures is 360°

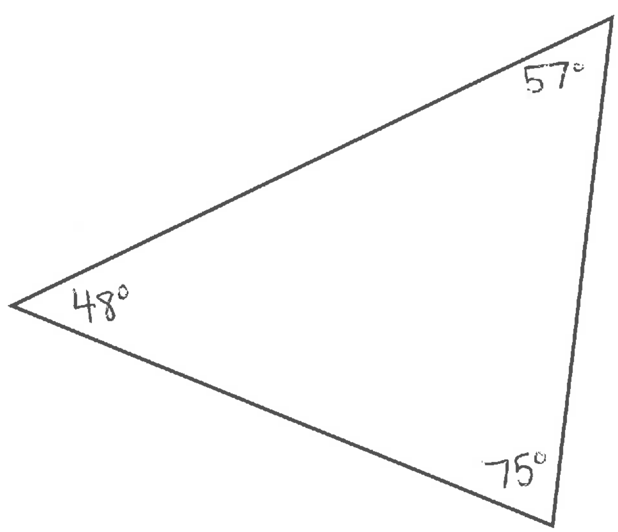
Pentagon:

The sum of the measures in 540°

My Notes







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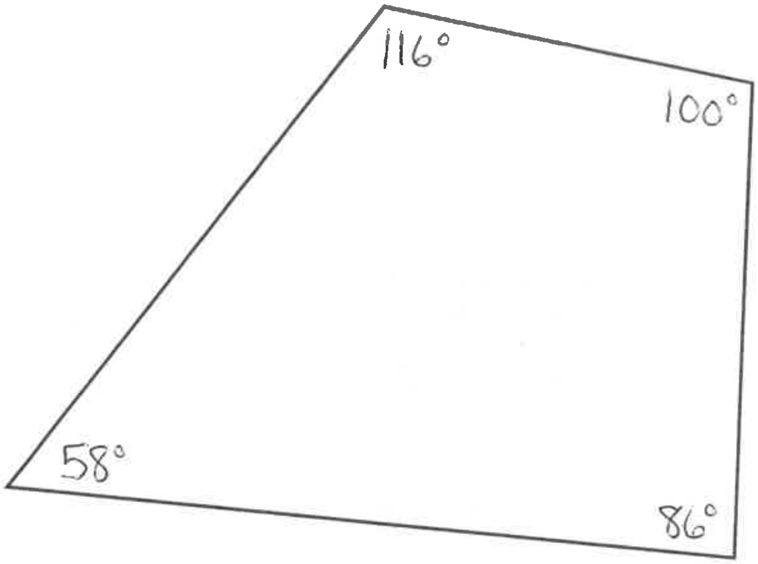
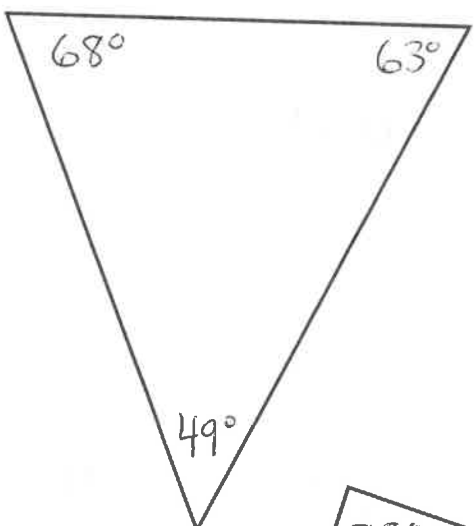
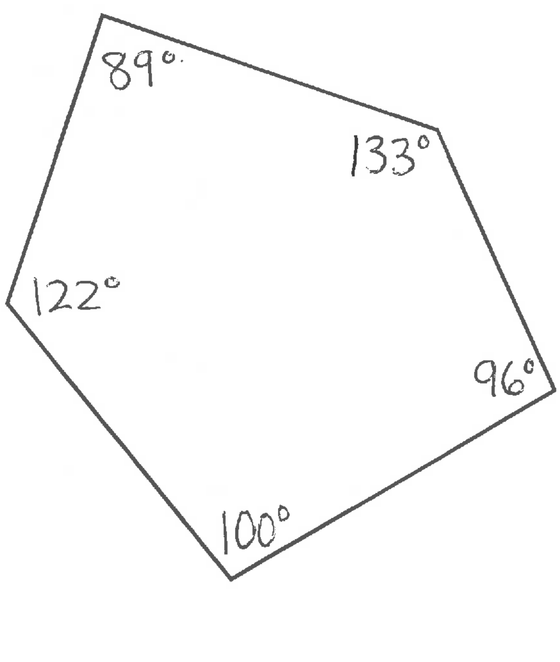






My Notes

ACADEMIC VOCABULARY

To ***replicate*** means to duplicate or imitate. Notice the word *replica* contained within the word. In science, experiments are designed so that they can be replicated by other scientists.

Sum of interior angles
$(n-2) \cdot 180°$

Knowing that the sum of the measures of the interior angles of a triangle is a constant and that the sum of the measures of nonoverlapping angles around a single point is always 360°, you can determine the sum of the measures of any polygon without measuring.

4. Use auxiliary segments to determine a way to predict and verify the exact sum of the angles of any quadrilateral and any pentagon. Describe your methods so that another group would be able to ***replicate*** your results for the pentagon.

Draw diagonals from a vertex to form triangles. In a quadrilateral, two triangles are formed; $180° \cdot 2 = 360°$

5. Use the method you described in Item 4 to answer the following.

a. Explain how to determine the sum of the measures of the interior angles of a hexagon.

In a hexagon, there are four triangles; $180° \cdot 4 = 720°$

b. Explain how to determine the sum of the measures of the interior angles for any polygon.

Use the formula $(n-2) \cdot 180°$

6. Express regularity in repeated reasoning. Use the method described in your answer to Item 5 to complete the table below.

Polygon	Number of Sides	Calculations	Sum of the Measures of the Interior Angles
Triangle	3	$(3-2) \cdot 180°$	180°
Quadrilateral	4	$(4-2) \cdot 180°$	360°
Pentagon	5	$(5-2) \cdot 180°$	540°
Hexagon	6	$(6-2) \cdot 180°$	720°
Heptagon	7	$(7-2) \cdot 180°$	900°
Octagon	8	$(8-2) \cdot 180°$	1080°
Nonagon	9	$(9-2) \cdot 180°$	1260°
Decagon	10	$(10-2) \cdot 180°$	1440°
Dodecagon	12	$(12-2) \cdot 180°$	1800°
n-gon	*n*	$(n-2) \cdot 180°$	

My Notes

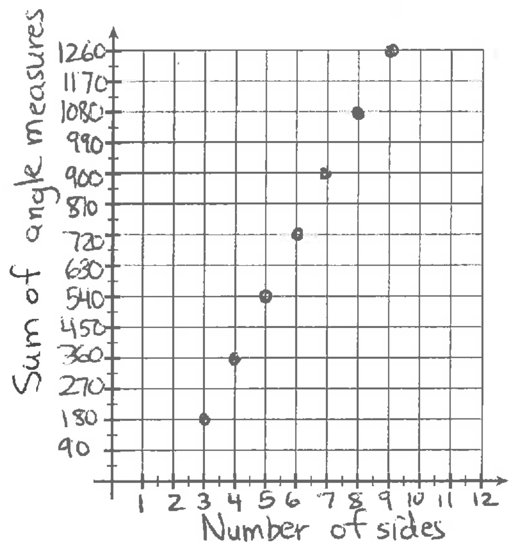
7. Observe in Item 6 that as the number of sides increases by one, the sum of the angle measures also increases by a constant amount. What type of function models this behavior?

The rate of change is constant; the function model is linear

8. For the first six polygons in Item 6, plot the ordered pair (number of sides, sum of angle measures) on the axes below. Carefully choose and label your scale on each axis.

CONNECT TO TECHNOLOGY

Use your graphing calculator to graph the ordered pairs. Create a table and then plot the points. Determine an appropriate scale for the axes.



9. The data points you graphed above should appear collinear. Write an equation for the line determined by these points.

$S=(n-2)\cdot 180$ or $S=180n-360$

10. **Make sense of problems.** State the numerical value of the slope of the line in Item 9 and describe what the slope value tells about the relationship between the number of sides and the sum of the measures of the interior angles of a polygon. Use units in your description.

The slope is 180°. The slope tells us that each time a side is added to a polygon, the sum of the measures of the interior angles of a polygon increases by 180°.

11. If the function S represents the sum of the measures of the interior angles as a function of the number of sides, n, write an algebraic rule for $S(n)$.

$S(n)=180n-360$

MATH TIP

Write the equation in slope-intercept form, $y = mx + b$.

12. Compare how the algebraic rule in Item 11 is similar to the slope-intercept form of a linear equation.

$S(n)=180n-360$

$m=180$

$b=-360$

My Notes

Each interior angle in a regular polygon

$\frac{(n-2)\cdot 180°}{n}$

13. Use $S(n)$ to determine the value of $S(7.5)$. For this value, explain the significance, if any, given that $S(n)$ represents the sum of the angle measures of an n-sided polygon.

$S(7.5) = 180 \cdot 7.5 = 990°$

The computed value doesn't make sense since a figure can't have 7.5 sides.

14. What is the domain of $S(n)$?

The domain is the set of all whole numbers for n greater than or equal to three.

15. How can you tell by looking at the graph in Item 8 that the function is linear? Explain why the domain is restricted.

MATH TERMS

Linear functions have *continuous* domains consisting of all real numbers. However, some contexts restrict the domain of linear functions. If the graph of a linear model consists of *discrete* points, the linear function is said to have a **discrete domain**.

Check Your Understanding

16. Determine the sum of the interior angles of a 20-sided polygon.

17. What is the measure of each interior angle of a regular polygon with 60 sides?

18. The sum of the interior angles of a polygon is 2340°. How many sides does the polygon have?

16) $(20-2)\cdot 180° = 3240°$

17) $\frac{(60-2)\cdot 180}{60} = 174°$

18) $\frac{(n-2)\cdot 180}{180} = \frac{2340}{180}$

$n-2 = 13$

$n = 15$ $\boxed{15 \text{ sides}}$

LESSON 31-1 PRACTICE

19. Does the rule in Item 11 hold true for various types of triangles? Create a way to prove the rule for a scalene triangle, a right triangle, and an isosceles triangle. Start by drawing one of the triangles, tearing it into three pieces, and rearranging the vertices.

20. **Critique the reasoning of others.** Denisha states that if you are given the sum of the interior angles of a polygon, there is no way to determine the number of sides of the polygon. Do you agree with Denisha? Justify your reasoning.

21. Use technology to create a hexagon. Label all angle measures but one. Exchange figures with a partner and find the missing angle measures in each hexagon using the rule you created in Item 11. Check answers using technology.

22. The angle measures of a stop sign are all the same. Determine the measure of each angle in a stop sign.

19) The rule holds true for any type of triangle.

20) No. Use $n = \frac{S(n)+360}{180}$

22) A stop sign is an octagon

$\frac{(8-2)\cdot 180}{8} = 135°$ each

My Notes

Learning Targets:

- Develop a formula for the measure of each interior angle of a regular polygon.
- Determine the measure of the exterior angles of a polygon.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Create Representations, Look for a Pattern, Quickwrite

A ***regular polygon*** is both ***equilateral*** and ***equiangular***. This means that each interior angle of a regular polygon has the same angle measure.

MATH TERMS

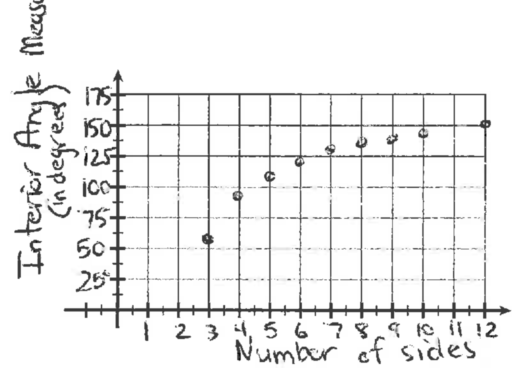
Equiangular means that all angles of a polygon are congruent. **Equilateral** means all sides of a polygon are congruent.

1. Complete the table to determine the angle measure of each interior angle for each regular polygon listed.

Polygon	Number of Sides	Sum of the Measures of the Interior Angles (degrees)	Measures of Each Interior Angle (degrees)
Triangle	3	180°	60°
Quadrilateral	4	360°	90°
Pentagon	5	540°	108°
Hexagon	6	720°	120°
Heptagon	7	900°	$128\frac{4}{7}°$
Octagon	8	1080°	135°
Nonagon	9	1260°	140°
Decagon	10	1440°	144°
Dodecagon	12	1800°	150°
n-gon	n	$180(n-2)°$	$\frac{180(n-2)°}{n}$

My Notes

2. For each regular polygon listed in the table in Item 1, plot the ordered pair (number of sides, measure of each interior angle) on the axes below. Carefully choose and label your scale on each axis.



3. **Make use of structure.** The points plotted in Item 2 should not appear collinear. Explain how that conclusion could have been drawn from the data alone.

The change in degrees isn't constant. They are not collinear

4. If the function E represents the measure of each interior angle as a function of the number of sides, n, write an algebraic rule for $E(n)$.

$$E(n) = \frac{180(n-2)}{n}$$

5. As n becomes greater, what appears to be happening to the measure of each angle? What causes this behavior?

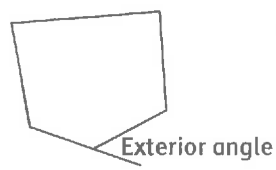
As the number of sides becomes greater, the angle measure increases and approaches 180° but doesn't get there since that would create a straight line rather than a polygon.

CONNECT TO TECHNOLOGY

Use the TABLE or GRAPHING component of a graphing calculator. Enter the algebraic function for $E(n)$ in y_1 to explore the measure of individual angle measures of a regular polygon as the number of sides increases.

My Notes

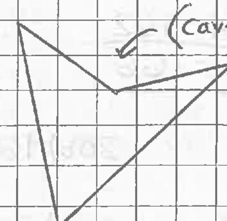
Exterior angles of a polygon also have a relationship. An ***exterior angle*** of a polygon is the angle formed between one side of a polygon and an extension of an adjacent side.



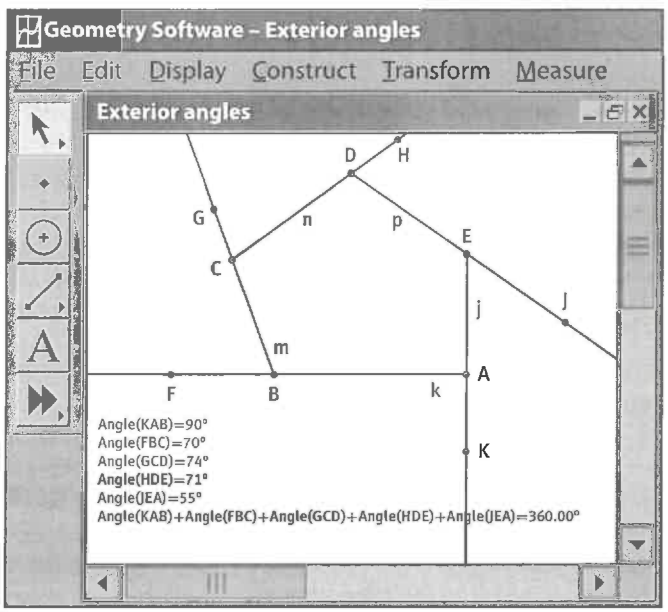
6. Use geometry software to draw a ***convex*** pentagon. Extend each side.

MATH TERMS

A polygon is **convex** if no line containing a side of the polygon contains a point in the interior of the polygon. A **convex polygon** has no interior angles greater than 180°.



7. Use appropriate tools strategically. Use the software to measure the exterior angles at each vertex. To do this you will need to mark a point on each ray.



8. Find the sum of the exterior angles from Item 7. 360°

9. Manipulate the polygon to form a convex hexagon. Measure each exterior angle and find the sum of the measures. 360°

10. Manipulate the polygon to form a convex heptagon. Measure each exterior angle and find the sum of the measures. 360°

11. Make a conjecture about the sum of the exterior angles of a polygon with *n* sides.

The sum of the exterior angles of a polygon with n sides is 360°

12. Predict the sum of the measure of the interior angle of a pentagon and one of its exterior angles. On what basis do you make your prediction? Use the software to measure and confirm your prediction.

The sum of the interior angle and the exterior angle is 180°; the angles form a linear pair, or a straight line.

My Notes

14) interior $=(24-2)\cdot 180$
$=3960°$
each interior $=165°$

15) It does not matter if the polygon is a regular polygon or not.

16) $125°$; $(540°-70-105-28-32)$

17) $\frac{360°}{24} = 15$ sides

18a) $2340°$ 18b) $2880°$

19) $\frac{(20-2)\cdot 180}{20} = 162$
$a-4=162$
$\boxed{a=166}$

20a) $156°$ 20b) $160°$

21a) $360°$ 21b) $360°$

22) $\frac{360}{18} = 20°$ $n+3=20$
$\boxed{n=17}$

23) Because 360 is not divisible by 80.

13. Use your answer from Item 12 and what you know about the angle sum of interior angles to further prove your conjecture from Item 11.

interior angle sum + exterior angle sum = 180n
180(n−2) + exterior angle sum = 180n
exterior angle sum = 180n − 180(n−2)
= 360°

Check Your Understanding

14. What is the sum of the interior angles of a regular 24-gon? What is the measure of each interior angle?

15. How does the sum of the measure of the exterior angles of a convex polygon differ if the polygon is a regular polygon? Explain.

16. A convex pentagon has exterior angles that measure 70°, 105°, 28°, and 32°. What is the measure of the fifth exterior angle?

17. Each exterior angle of a regular polygon measures 24°. How many sides does the polygon have?

LESSON 31-2 PRACTICE

18. Find the sum of the measures of the interior angles of each regular polygon.
a. 15-gon **b.** 18-gon

19. The expression $(a-4)°$ represents the measure of an interior angle of a regular 20-gon. What is the value of a in the expression?

20. Find the measure of each interior angle of each regular polygon.
a. 15-gon **b.** 18-gon

21. Find the sum of the measures of the exterior angles of each polygon.
a. 15-gon **b.** 18-gon

22. The expression $(n+3)°$ represents the measure of an exterior angle of a regular 18-gon. What is the value of n in the expression?

23. Construct viable arguments. Emilio said that he drew a regular polygon using a protractor and measured one of its interior angles as 100°. Explain, using what you know about exterior angles, why this cannot be true.

24. Use graphing software to draw three quadrilaterals: a square, a rectangle, and a nonequiangular quadrilateral.
a. Find the measure of each interior angle of each polygon.
b. Prove that the sum of the measures of the interior angles of each quadrilateral is the same.
c. Prove that the sum of the measures of the exterior angles of each quadrilateral is 360°.

My Notes

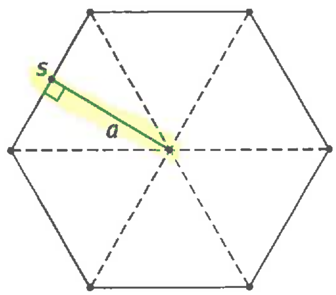
NOTE: The apothem DOES NOT extend from the middle to a vertex.

Learning Targets:

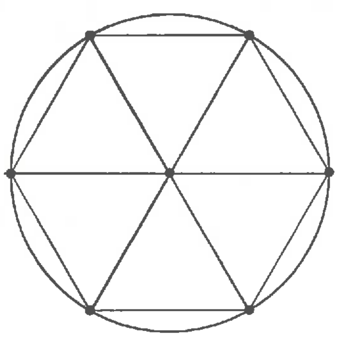
- Develop a formula for the area of a regular polygon.
- Solve problems using the perimeter and area of regular polygons.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Quickwrite, Create Representations

Recall that a regular polygon is equilateral and equiangular. An ***apothem*** of a regular polygon is a perpendicular segment from a midpoint of a side of a regular polygon to the center of the circle circumscribed about the polygon. The apothem of a regular hexagon of side s is indicated by a in the diagram.



1. Consider the tabletop template of the regular hexagon shown below.



a. Draw all radii from the center of the circumscribed circle to each vertex of the hexagon.

b. What is the measure of each of the central angles formed by the radii? Explain how you arrived at your answer. 60°; A circle has 360°; there are six central angles $\frac{360°}{6} = 60°$

c. Classify the triangles formed by the radii by their side length. Measure to verify your answer. equilateral triangles

d. How is the apothem of the hexagon related to the triangles formed by the radii? It is the height of the equilateral triangle.

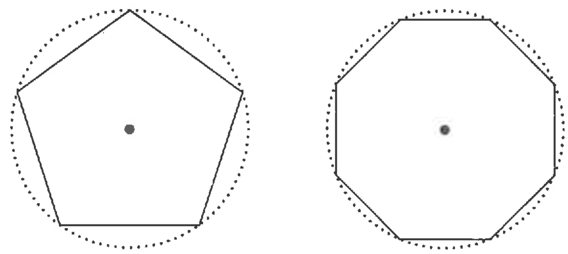
e. Find the area of a regular hexagon with side length 4 ft. Show the calculations that lead to your answer.

$24\sqrt{3}$ ft^2 ; $\underset{\text{6 triangles}}{6}\left(\frac{1}{2}\underset{\text{base}}{(4)}\underset{\text{height}}{(2\sqrt{3})}\right)$

My Notes

Now you can use what you discovered in Item 1 to generalize the formula to determine the area of any regular polygon.

2. **Express regularity in repeated reasoning.** Refer to the diagrams of the regular polygons shown.



a. How many triangles are formed when the radii are drawn from the center of the polygon to each of the vertices of the polygon with n sides? n

b. What is the measure of each of the central angles formed by the radii of the n-gon? Explain how you arrived at your answer.

$\frac{360°}{n}$; divide 360° by n radii

c. Classify the triangles formed by the radii by their side length. Verify your answer.

isosceles triangles; the radii of the circle are congruent

d. Write an expression that finds the area of one triangle in an n-gon, using a to represent the apothem and s to represent a side of the n-gon. Then use the expression to write a formula that finds the area of the entire n-gon, where n represents the number of sides of the n-gon.

$\frac{1}{2}as$; $A = \frac{1}{2}as \cdot n$

e. Explain how to determine the perimeter of a polygon, P.

Multiply the side lengths, s, by the number of sides

f. Write a formula that can be used to calculate the area of a regular n-gon with apothem length a and perimeter P, by substituting P in the formula you wrote in part d.

$A = \frac{1}{2}aP$

My Notes

Example A

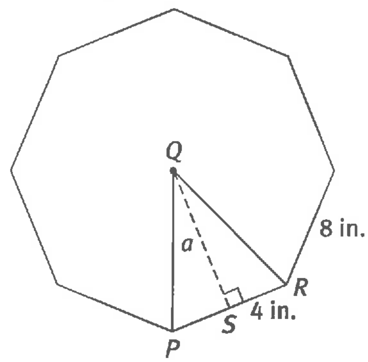
Calculate the area of a regular octagon with side length 8 inches.

Step 1: Find the apothem.

Draw the apothem, $\overline{QS}$.
The length of $\overline{QS}$ is a.

Since the polygon is a regular octagon,
$m\angle PQR = \frac{360^\circ}{8} = 45^\circ$.

So, $m\angle SQR = 22.5^\circ$.



Use trigonometry to find a:
$\frac{4}{a} = \tan 22.5^\circ$, so $a \approx 9.7$ in.

Step 2: Find the area.

$$A = \frac{1}{2}aP$$
$$= \frac{1}{2}(9.7\text{ in.})(8)(8\text{ in.})$$
$$= 310.4\text{ in.}^2$$

Solution: The area is approximately 310.4 square inches.

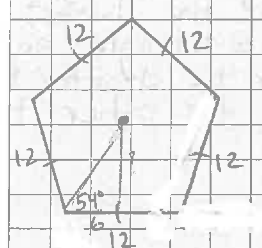
Try These A

a. Calculate the area of a regular pentagon with side length 12 m.

$A = \frac{1}{2}(6 \cdot \tan 54^\circ)(60)$

$A \approx 247.75\text{ m}^2$

a)



$\tan 54^\circ = \frac{a}{6}$

$a = 8.25829$

$P = 60$

b. Find the area of a regular hexagon with 18 cm sides.

$a = 9\sqrt{3}$ cm $\quad A = \frac{1}{2}(9\sqrt{3})(108)$

$P = 108$ cm $\quad A \approx 841.78\text{ cm}^2$

c. Reason quantitatively. Find the perimeter and area of regular polygon $ABCD$ with vertices $A(7, 8)$, $B(9, -1)$, $C(0, -3)$, and $D(-2, 6)$.

perimeter $= 4\sqrt{85}$ units

area $= 85$ units^2

My Notes

3) $30°-60°-90°$ right triangle

4) $P = sn$; s = side length, n = number of sides

5). $a = (\tan 67.5°)(3.5)$
$P = 56$ inches
$A = 236.6 \text{ in}^2$

6) Yes; the radius is the hypotenuse of the right triangle and the apothem is the leg of the triangle. The hypotenuse is always the longest side of a triangle, so it can never be shorter than either of the other two sides.

8) $P = 320$ inches
$a = (20)(\tan 67.5°)$
$A = 7726 \text{ in}^2$

9) 25 units^2

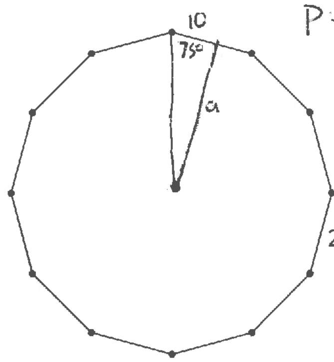
10) 152 cm^2

Check Your Understanding

3. The properties of what type of triangle can be used to determine the apothem in a regular hexagon?
4. Write a formula that can be used to determine the perimeter of a regular n-gon.
5. Find the area of a regular octagon with 7-inch sides.
6. Seth states that the radius of a regular polygon is never less than its apothem. Do you agree? If so, provide justification. If not, provide a counterexample.

LESSON 31-3 PRACTICE

7. Determine the perimeter and area of the regular dodecagon.



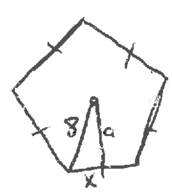
$P = 240\text{cm}$

$\tan 75° = \frac{a}{10}$

$A = \frac{1}{2}(10 \cdot \tan 75°)(240)$

$A = 4479 \text{ cm}^2$

8. **Make sense of problems.** A manufacturing company is making a design for an octagonal hazard sign. Each side of the sign measures 40 in. What is the area of the sign?
9. Find the area of regular polygon $QRST$ with vertices at $Q(1, 3)$, $R(5, 0)$, $S(2, -4)$, and $T(-2, -1)$.
10. Find the area of a regular pentagon with a radius of 8 cm.



$\cos 54° = \frac{x}{8}$

$x = 4.70228$

$P = 75.2365\text{cm}$

$A = 152 \text{cm}^2$